Synthesis of Helicopter Stabilization Systems Using Modal Control Theory

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This paper presents procedures for designing controllers for complex dynamical systems using single- and multi-input modal control theory. It is shown that the design of the appropriate modal controller for a given system is greatly facilitated by the inspection of mode-controllability indices which constitute the mode-controllability matrix of the system. The design procedures enable controllers for multi-input systems to be synthesized by sequential applications of a single-input theory or by an alternative method in which it is possible to impose gain constraints on the modal controller. The power of these procedures is demonstrated by the design of both longitudinal and lateral controllers for the Sikorsky SH-3D Sea King helicopter.

Nomenclature

$\mathbf{A}$ = $n \times n$ matrix of uncontrolled system
$\mathbf{B}$ = $n \times r$ input matrix
$\mathbf{b}_j$ = jth column of $\mathbf{B}$
$\mathbf{C}$ = $n \times n$ matrix of controlled system
$\mathbf{F}_i$ = matrix defined in Eq. (20)
$\mathbf{G}$ = $r \times n$ gain matrix
g_j' = jth row of $\mathbf{G}$
i, j, k = integers
K_{ij} = proportional controller gain
l, m, n = integers
$\mathbf{P}$ = mode-controllability matrix
p_{ji} = mode-controllability index
p = roll rate (rad/sec)
$\mathbf{q}_i$ = vector defined in Eq. (20)
q = pitch rate (rad/sec)
r = integer
r = yaw rate (rad/sec)
s_j = signal
U_c = main-rotor collective pitch-control angle (rad)
U_p = longitudinal cyclic pitch-control angle (rad)
U_R = lateral cyclic pitch-control angle (rad)
U_T = tail-rotor collective pitch-control angle (rad)
$\mathbf{u}_j$ = $n \times 1$ eigenvector of $\mathbf{A}$
u = longitudinal velocity (ft/sec)
$\mathbf{V}_m$ = $n \times m$ matrix of m eigenvectors $\mathbf{v}_j$
V_R = rotor-tip velocity (ft/sec)
$\mathbf{V}$ = modal matrix of $\mathbf{A}'$
$\mathbf{v}_j$ = $n \times 1$ eigenvector of $\mathbf{A}'$
v = lateral velocity (ft/sec)
$\mathbf{w}_j$ = eigenvector of $\mathbf{C}$
w = vertical velocity (ft/sec)
$\mathbf{x}$ = $n \times 1$ state vector
$\mathbf{y}$ = $r \times 1$ input vector
δ_{ij} = Kronecker delta
θ = pitch angle (rad)
λ_j = eigenvalue of uncontrolled system
$\boldsymbol{\mu}_j$ = measurement vector
ν_j = integer
ρ_j = eigenvalue of controlled system
ϕ = roll angle (rad)
ψ = yaw angle (rad)

Superscripts

$'$ = transpose of a matrix
$(\ )^*$ = complex conjugate

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1. Introduction

THIS paper is concerned with the presentation of procedures for designing complex dynamical systems which are based on the use of modal control theory.[1–8] It is shown that, by the introduction of controllability indices into modal control theory, these design procedures can be rationalized in the following two principal respects. 1) The dynamical modes of a system which can be controlled by a given input may be readily identified. 2) If 1) indicates that it is possible to control a given set of modes by alternative sets of inputs, the choice of controls is greatly facilitated by inspection of the magnitudes of the appropriate controllability indices.

In addition, it is shown that the control laws derived on the basis of modal control theory are very simple to compute, since they can be defined by closed-form formulae. The particular design procedures described make it possible to design feedback loops which improve the response characteristics of a system by altering complex conjugate and real eigenvalues of an uncontrolled system simultaneously. Theories are presented for both multivariable, single- and multi-input systems: it is demonstrated that feedback loops for multi-input systems can be designed by sequential applications of the single-input theory or by an alternative method in which it is possible to impose gain constraints on the modal controller.

These procedures are shown to be directly applicable and particularly appropriate to the design of controllers for the mode-stabilization of helicopters since these aircraft often have large modal coupling and therefore usually present major design difficulties. Thus, for example, in the design of a lateral auto-stabilizer for a helicopter, the lateral modal controller could be designed using the lateral cyclic pitch-control to improve the characteristics of the Dutch roll and the tail-rotor collective pitch-control to improve the characteristics of the yawing and directional modes: alternatively, the lateral cyclic pitch-control and the tail-rotor collective pitch-control can together provide a degree of control over a number of the lateral modes.

These design procedures are illustrated in this paper by the detailed design of both longitudinal and lateral autostabilizers for the Sikorsky SH-3D Sea King helicopter.[9] This choice of aircraft enables a direct comparison of these procedures to be made with those based on optimal control theory given by Murphy and Narendra.[9]

2. General Theory of Modal Control

2.1 Multi-Input Theory

The general theory of multi-input modal control is concerned with the design of controllers for multivariable linear systems governed by state equations of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \tag{1}$$

where $\mathbf{x}$ is the $n \times 1$ state vector of the system, $\mathbf{A}$ is the $n \times n$ matrix of the uncontrolled system, $\mathbf{B}$ is the $n \times r$ input matrix and $\mathbf{y}$ is the $r \times 1$ control vector.

In this theory $\mathbf{A}$ may have several real and several pairs of conjugate complex eigenvalues. In addition, it is assumed that all the elements of the state vector $\mathbf{x}$ can be measured by appropriate transducers, and therefore that it will be possible to combine the transducer outputs to generate $m(m \leq n)$ signals given by equations of the form

$$s_j = \sum_{k=1}^{n} \mu_{jk} x_k = \boldsymbol{\mu}_j' \mathbf{x} (j = 1,2,\ldots,m) \tag{2}$$

where $\boldsymbol{\mu}_j$ is a measurement vector and the prime denotes matrix transposition. These new signals, s_j, can then be amplified by $r \times m$ proportional controllers having gains K_{ij} $(i = 1,2,\ldots, r;\ j = 1,2,\ldots, m)$ thus yielding control signals given by

$$y_i = \sum_{j=1}^{m} K_{ij} s_j = \sum_{j=1}^{m} K_{ij} \boldsymbol{\mu}_j' \mathbf{x} \quad (i = 1,2,\ldots,r) \tag{3}$$

It follows by substituting the expression for y_i given in Eq. (3) into Eq. (1) that the governing equation of the resulting closed-loop system is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{r} \mathbf{b}_i \sum_{j=1}^{m} K_{ij} \boldsymbol{\mu}_j' \mathbf{x} = \mathbf{C}\mathbf{x} \tag{4}$$

where the $\mathbf{b}_i$ are the columns of $\mathbf{B}$. Equation (4) indicates that the effect of the control signals defined by Eq. (3) is to change the system matrix $\mathbf{A}$ to a new matrix $\mathbf{C}$ given by

$$\mathbf{C} = \mathbf{A} + \sum_{i=1}^{r} \mathbf{b}_i \sum_{j=1}^{m} K_{ij} \boldsymbol{\mu}_j' \tag{5}$$

Now if $\mathbf{u}_j$ and $\mathbf{v}_j$ are respectively the jth eigenvectors of $\mathbf{A}$ and $\mathbf{A}'$, and λ_j is the associated eigenvalue‡, then

$$\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j \quad (j = 1,2,\ldots,n) \tag{6}$$

and

$$\mathbf{A}'\mathbf{v}_j = \lambda_j \mathbf{v}_j \quad (j = 1,2,\ldots,n) \tag{7}$$

where the two sets of eigenvectors satisfy orthogonality conditions of the form

$$\mathbf{v}_j'\mathbf{u}_k = 0 \quad (j \neq k; j, k = 1,2,\ldots,n) \tag{8}$$

and can be normalized so that

$$\mathbf{v}_j'\mathbf{u}_j = 1 \quad (j = 1,2,\ldots,n) \tag{9}$$

If $\boldsymbol{\mu}_j$ is chosen to be equal to $\mathbf{v}_j$, then the matrix of the controlled system given by Eq. (5) will have the particular form

$$\mathbf{C} = \mathbf{A} + \sum_{i=1}^{r} \mathbf{b}_i \sum_{j=1}^{m} K_{ij} \mathbf{v}_j' \tag{10}$$

It can now be deduced from Eqs. (8) and (10) that if

$$m + 1 \leq k \leq n$$

then

$$\mathbf{C}\mathbf{u}_k = \mathbf{A}\mathbf{u}_k = \lambda_k \mathbf{u}_k \tag{11}$$

thus indicating that $\mathbf{u}_k$ and λ_k $(k = m + 1,\ m + 2,\ldots,n)$ are eigenvectors and eigenvalues of $\mathbf{C}$ as well as of $\mathbf{A}$. It is also evident from Eqs. (8) and (10) that if

$$1 \leq j \leq m$$

then

$$\mathbf{C}\mathbf{u}_j = \mathbf{A}\mathbf{u}_j + \sum_{i=1}^{r} K_{ij} \mathbf{b}_i = \lambda_j \mathbf{u}_j + \sum_{i=1}^{r} K_{ij} \mathbf{b}_i \tag{12}$$

which implies that, in general, λ_j is not an eigenvalue of $\mathbf{C}$ and also that $\mathbf{u}_j$ is no longer the corresponding eigenvector if $K_{ij} \neq 0$. The effect of using m measurement vectors is thus to change the set of eigenvalues $\{\lambda_j\}$ to a set of new values $\{\rho_i\}$ and the set of eigenvectors $\{\mathbf{u}_j\}$ to a corresponding set of new vectors $\{\mathbf{w}_j\}$ $(j = 1,2,\ldots, m)$ leaving the remaining sets of eigenvalues $\{\lambda_k\}$ and eigenvectors $\{\mathbf{u}_k\}$ $(k = m + 1,\ m + 2,\ \ldots, n)$ of the system matrix unchanged.

In order to calculate values of the proportional controller gains, K_{ij}, in terms of the eigenvalues of $\mathbf{A}$ and the required eigenvalues of $\mathbf{C}$, let

$$\mathbf{w}_j = \sum_{k=1}^{n} q_{jk} \mathbf{u}_k \qquad (j = 1,2,\ldots,m) \tag{13}$$

and let

$$\mathbf{b}_i = \sum_{j=1}^{n} p_{ij} \mathbf{u}_j \qquad (i = 1,2,\ldots,r) \tag{14}$$

where the $\mathbf{u}_j$ are linearly independent vectors and may thus be used as a basis of an n-dimensional vector space. It follows from Eqs. (8, 9 and 14) that

$$p_{ij} = \mathbf{v}_j'\mathbf{b}_i = \mathbf{b}_i'\mathbf{v}_j \qquad (i = 1,2,\ldots,\ r;\ j = 1,2,\ldots n) \tag{15}$$

Since, by definition, ρ_j and $\mathbf{w}_j$ satisfy the equations

$$\mathbf{C}\mathbf{w}_j = \rho_j \mathbf{w}_j \qquad (j = 1,2,\ldots,m) \tag{16}$$

substituting from Eqs. (10) and (13) into Eq. (16) yields the equations

$$\left(\mathbf{A} + \sum_{l=1}^{r} \mathbf{b}_l \sum_{j=1}^{m} K_{lj} \mathbf{v}_j'\right) \sum_{k=1}^{n} q_{ik} \mathbf{u}_k = \rho_i \sum_{k=1}^{n} q_{ik} \mathbf{u}_k \qquad (i = 1,2,\ldots,m) \tag{17}$$

Equations (6, 8, 9 and 14) imply that Eq. (17) may be expressed in the form

$$\sum_{k=1}^{n} q_{ik} \lambda_k \mathbf{u}_k + \sum_{l=1}^{m} \sum_{k=1}^{n} \left(p_{lk} \mathbf{u}_k \sum_{j=1}^{m} K_{lj} q_{ij}\right) = \rho_i \sum_{k=1}^{n} q_{ik} \mathbf{u}_k \qquad (i = 1,2,\ldots,m) \tag{18}$$

Equation (18) is a vector equation in the $\mathbf{u}_k$ and is equivalent to the $(n \times m)$ scalar equations

$$(\rho_i - \lambda_k) q_{ik} - \sum_{l=1}^{r} p_{lk} \sum_{j=1}^{m} K_{lj} q_{ij} = 0 \qquad (i = 1,2,\ldots,m;\ k = 1,2,\ldots,n) \tag{19}$$

It is evident that, for a given value of i, the first m of Eqs. (19) (i.e., those involving the eigenvalues to be altered, $\lambda_1, \lambda_2, \ldots, \lambda_m$) can be written in the matrix form

$$\mathbf{F}_i \mathbf{q}_i = \mathbf{0} \qquad (i = 1,2,\ldots,m) \tag{20a}$$

where

$$\mathbf{F}_i = [f_{jk}^{(i)}] \tag{20b}$$

$$f_{jk}^{(i)} = (\rho_i - \lambda_k)\delta_{jk} - \sum_{l=1}^{r} p_{lk} K_{lj} \quad (i,j,k = 1,2,\ldots,m) \tag{20c}$$

$$\mathbf{q}_i = [q_{ij}] \tag{20d}$$

‡ It is assumed throughout this paper that the eigenvalues λ_j are distinct.

and δ_{jk} is the Kronecker delta. Since $\mathbf{q}_i \neq \mathbf{0}$, it follows from Eq. (20a) that

$$\det \mathbf{F}_i = 0, \quad (i = 1,2,\ldots,m) \tag{21}$$

which, in view of Eq. (20b) and Eq. (20c), has the explicit form

$$\begin{vmatrix} \rho_i - \lambda_i - \sum_{j=1}^{r} p_{j1}K_{j1}, & -\sum_{j=1}^{r} p_{j1}K_{j2},\ldots, & -\sum_{j=1}^{r} p_{j1}K_{jm} \\ -\sum_{j=1}^{r} p_{j2}K_{j1}, & \rho_i - \lambda_2 - \sum_{j=1}^{r} p_{j2}K_{j2},\ldots, & -\sum_{j=1}^{r} p_{j2}K_{jm} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ -\sum_{j=1}^{r} p_{jm}K_{j1}, & -\sum_{j=1}^{r} p_{jm}K_{j2},\ldots, & \rho_i - \lambda_m - \sum_{j=1}^{r} p_{jm}K_{jm} \end{vmatrix} = 0 \quad (i = 1,2,\ldots,m) \tag{22}$$

2.2 Single-Input Theory

In the case of multivariable, multi-input systems the expansion of the determinant in Eq. (22) yields, for the jth control input acting alone, the equation

$$\sum_{k=1}^{m} (\rho_i - \lambda_k) = \sum_{l=1}^{m} K_{jl}p_{jl} \prod_{\substack{k=1\\k\neq i}}^{m} (\rho_i - \lambda_k) \quad (i = 1,2,\ldots,m) \tag{23}$$

which implies that

$$\sum_{k=1}^{m} \frac{K_{jk}p_{jk}}{(\rho_i - \lambda_k)} = 1 \qquad (i = 1,2,\ldots,m) \tag{24}$$

provided that

$$\rho_i - \lambda_k \neq 0 \qquad (i,k = 1,2,\ldots,m)$$

Equations (24) can be solved for the K_{ji} to give the formulae[6]

$$K_{ji} = \frac{\prod_{k=1}^{m} (\rho_k - \lambda_i)}{p_{ji} \prod_{\substack{k=1\\k\neq i}}^{m} (\lambda_k - \lambda_i)} \qquad (i = 1,2,\ldots,m) \tag{25}$$

which indicate that the gains, K_{ji}, will be calculable since

$$\lambda_k \neq \lambda_i \qquad (i,k = 1,2,\ldots,m)$$

provided that

$$p_{ji} \neq 0 \quad (i = 1,2,\ldots,m) \tag{26}$$

The single-input law obtained by substituting from Eq. (25) into Eq. (4), and also by putting $\boldsymbol{\mu}_j = \mathbf{v}_j$ in this equation, has the form

$$y_j = \sum_{i=1}^{m} \left[\prod_{k=1}^{m} (\rho_k - \lambda_i)/p_{ji} \prod_{\substack{k=1\\k\neq i}}^{m} (\lambda_k - \lambda_i) \right] \quad \mathbf{v}_i'\mathbf{x} = \mathbf{g}_j'\mathbf{x} \tag{27}$$

This control law will alter the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_m$, of the matrix $\mathbf{A}$ of the uncontrolled system to prescribed new values $\rho_1,\rho_2,\ldots,\rho_m$, leaving the remaining $(n-m)$ eigenvalues λ_{m+1}, λ_{m+2}, λ_{m+3}, $\lambda_{m+4},\ldots,\lambda_n$, unaltered. It is important to note that, in the case of real systems, the control law defined in Eq. (27) will be real, even in the case of complex eigenvalues. This follows from the fact that if $\lambda_k = \lambda_i^*$ then $\mathbf{v}_k = \mathbf{v}_i^*$ and $p_{jk} = p_{ji}^*$, together with the fact that the required new eigenvalues will either occur in conjugate complex pairs or be real.

3. Multi-Input Design Procedures

3.1 Basis of Design Procedures

The formulae developed in Sec. 2 of this paper make it possible to design feedback loops which improve the response characteristics of a system by altering selected eigenvalues of the system matrix: these eigenvalues may be real or complex conjugates. In the case of many multi-input systems a given mode can be controlled by more than one input so that it is possible to control such a mode by choosing only one, or a selection of, these inputs. In order to rationalize the selection of the appropriate inputs to control a given set of modes, it is necessary to recall the properties of the p_{ji} that occur in Eq. (27): the p_{ji} are the elements of the matrix

$$\mathbf{P} = \mathbf{B}'\mathbf{V} \tag{28}$$

where $\mathbf{B}$ is the $n \times r$ input matrix and $\mathbf{V}$ is the $n \times n$ modal matrix of $\mathbf{A}'$. It is well-known that if $p_{ji} \neq 0$ then the ith mode of the system can be controlled by the jth input. Since in Eq. (27) the p_{ji} occur in the denominator of the expression for the gain vector $\mathbf{g}_j'$, it is evident that this choice will normally be made by selecting the jth input to control the ith mode in accordance with the inequality.

$$|p_{ji}| > |p_{ki}| \quad (k \neq j, k = 1,2,\ldots,r; \quad i = 1,2,\ldots,n) \tag{29}$$

The matrix $\mathbf{P}$ in Eq. (28) is accordingly termed the mode-controllability matrix and its elements, p_{ji}, the mode-controllability indices.

3.2 Application of Single-Input Theory

In the case of a single-input system, it is first necessary to identify the controllable and uncontrollable modes on the basis of Eq. (28). The eigenvalues associated with the controllable modes can then be assigned any desired values by computing a feedback-control law in accordance with Eq. (27).

3.3 Sequential Application of Single-Input Theory

The theory for single-input systems presented in Sec. 2.2 can be used to design feedback loops for multi-input systems by first writing the state equation (1) in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{j=1}^{r} \mathbf{b}_j y_j \tag{30}$$

It is evident that each of the y_j in Eq. (30) can be generated in the manner described in Sec. 3.2 thus forming a sequence of inputs $\{y_\alpha, y_\beta, y_\gamma, \ldots\}$ where the set of modes $\{m_{\nu 1}, m_{\nu 2}, \ldots\}$ controlled by a given input, y_ν, is selected in accordance with the design basis described in Sec. 3.1.

3.4 Application of Multi-Input Theory with Gain Constraints

An alternative method of designing multi-input modal controllers to that described in Sec. 3.3 can be developed on the basis of the m Eqs. (22) which involve the $m \times r$ quantities K_{ji} $(j = 1,2,\ldots,r;\ i = 1,2,\ldots,m)$. Since these equations obviously do not determine a unique set of the K_{ji} in the case $r > 1$, it is possible to introduce additional design criteria if desired.

Thus, for example, in the case of a closed-loop system with m modes controlled by r inputs, the system matrix may be written in the form

$$\mathbf{C} = \mathbf{A} + \mathbf{B}\mathbf{G} \tag{31}$$

where

$$\mathbf{G} = \mathbf{K}\mathbf{V}_m' \tag{32}$$

$$\mathbf{K} = [K_{ji}]\ (j = 1,2,\ldots,r;\quad i = 1,2,\ldots,m) \tag{33}$$

and

$$\mathbf{V}_m = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m] \tag{34}$$

It follows from Eq. (32) that the gain of the jth input with respect to the ith state variable is given by

$$g_{ji} = \sum_{k=1}^{m} K_{jk}v_{ik} \quad (j = 1,2,\ldots,r;\quad i = 1,2,\ldots,n) \tag{35}$$

Equation (35) implies that even if the ith state variable is not available for feedback, then it may nevertheless be possible to choose any desired set of closed-loop eigenvalues by ensuring that the K_{ji} satisfy the equations

$$\sum_{k=1}^{m} K_{jk}v_{ik} = 0 \quad (j = 1,2,\ldots,r) \tag{36}$$

This design approach thus obviates the need for incorporating a state observer to estimate the state variables that are not available for feedback.

4. Design Examples

4.1 Uncontrolled System

The modal control design procedures developed in this paper can be conveniently illustrated by using them to synthesize appropriate feedback loops for the stabilization of the Sikorsky SH-3D Sea King helicopter in hovering flight. The mathematical model of the hover dynamics has been described in detail by Murphy and Narendra[9]: the state equation of the uncontrolled system has the form

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}\\ \mathbf{A}_{21}, & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} + \begin{bmatrix}\mathbf{B}_{11}, & \mathbf{B}_{12}\\ \mathbf{B}_{21}, & \mathbf{B}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} \tag{37}$$

In Eq. (37) the longitudinal state vector $\mathbf{x}_1$ and the lateral state vector $\mathbf{x}_2$ are respectively given by

$$\mathbf{x}_1 = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}u/V_R\\ w/V_R\\ q\\ \theta\end{bmatrix} \tag{38a}$$

and

$$\mathbf{x}_2 = \begin{bmatrix}x_5\\ x_6\\ x_7\\ x_8\\ x_9\end{bmatrix} = \begin{bmatrix}v/V_R\\ p\\ r\\ \phi\\ \psi\end{bmatrix} \tag{38b}$$

In addition, the longitudinal control vector $\mathbf{y}_1$ and the lateral control vector $\mathbf{y}_2$ are respectively given by

$$\mathbf{y}_1 = \begin{bmatrix}U_P\\ U_C\end{bmatrix} \tag{39a}$$

and

$$\mathbf{y}_2 = \begin{bmatrix}U_R\\ U_T\end{bmatrix} \tag{39b}$$

In order to illustrate the design procedures as clearly as possible, the longitudinal and lateral modes will be decoupled by assuming that the coupling matrices $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{B}_{12}$ and $\mathbf{B}_{21}$ in Eq. (37) are all null. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{B}_{11}$ and $\mathbf{B}_{22}$ (as given by Murphy and Narendra[9]) are

$$\mathbf{A}_{11} = \begin{bmatrix}-0.016, & 0, & +0.0025, & -0.05\\ 0, & -0.3242, & +0.0002, & 0\\ +1.97, & +1, & -0.542, & 0\\ 0, & 0, & +1, & 0\end{bmatrix} \tag{40a}$$

$$\mathbf{A}_{22} = \begin{bmatrix}-0.033, & -0.0025, & +0.0009, & +0.05, & 0\\ -7.25, & -1.96, & +0.01, & 0, & 0\\ +5.59, & -0.0043, & -0.303, & 0, & 0\\ 0, & +1, & 0, & 0, & 0\\ 0, & 0, & +1, & 0, & 0\end{bmatrix} \tag{40b}$$

$$\mathbf{B}_{11} = \begin{bmatrix}+0.005, & +0.05\\ -0.424, & 0\\ +0.69, & -6.15\\ 0, & 0\end{bmatrix} \tag{40c}$$

and

$$\mathbf{B}_{22} = \begin{bmatrix}+0.05, & +0.022\\ +21.81, & +0.3475\\ +0.174, & -7.48\\ 0, & 0\\ 0, & 0\end{bmatrix} \tag{40d}$$

The eigenvalues of $\mathbf{A}_{11}$ are

$$\lambda_1 = +0.0887 + 0.3552i, \quad \lambda_2 = +0.0887 - 0.3552i \tag{41}$$
$$\lambda_3 = -0.7354, \quad \lambda_4 = -0.3240$$

and the corresponding eigenvectors of $\mathbf{A}_{11}'$ are

$$\mathbf{v}_1 = \begin{bmatrix}+1.0000\\ +0.2898 + 0.1873i\\ +0.0531 + 0.1803i\\ -0.0331 + 0.1325i\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}+1.0000\\ +0.2898 - 0.1873i\\ +0.0531 - 0.1803i\\ -0.0331 - 0.1325i\end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix}+0.7205\\ +0.6398\\ -0.2631\\ +0.0490\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}-0.0009\\ +1.0000\\ +0.0001\\ -0.0001\end{bmatrix} \tag{42}$$

In accordance with Eq. (28), the longitudinal mode-controllability matrix is

$$\mathbf{P}_1' = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]'\,\mathbf{B}_{11}$$

$$= \begin{bmatrix}-0.2767 - 1.1087i, & -0.0812 + 0.0450i\\ -0.2767 + 1.1087i, & -0.0812 - 0.0450i\\ +1.6541 & , -0.4492\\ -0.0010 & , -0.4239\end{bmatrix} \tag{43}$$

The eigenvalues of $\mathbf{A}_{22}$ are

$$\lambda_5 = +0.0315 + 0.4137i, \quad \lambda_6 = +0.0315 - 0.4137i$$
$$\lambda_7 = -2.0565, \quad \lambda_8 = -0.3024, \quad \lambda_9 = 0 \tag{44}$$

and the corresponding eigenvectors of $\mathbf{A}_{22}'$ are

$$\mathbf{v}_5 = \begin{bmatrix}+1.0000\\ -0.0088 - 0.0585i\\ +0.0001 - 0.0019i\\ +0.0091 - 0.1202i\\ 0\end{bmatrix}$$

$$\mathbf{v}_6 = \begin{bmatrix}+1.000\\ -0.0088 + 0.0585i\\ +0.0001 + 0.0019i\\ +0.0091 + 0.1202i\\ 0\end{bmatrix}$$

$$\mathbf{v}_7 = \begin{bmatrix}-0.9633\\ -0.2673\\ +0.0020\\ +0.0234\\ 0\end{bmatrix}, \quad \mathbf{v}_8 = \begin{bmatrix}+0.9670\\ -0.0975\\ -0.1730\\ -0.1599\\ 0\end{bmatrix}, \quad \mathbf{v}_9 = \begin{bmatrix}0\\ +0.3866\\ +0.5013\\ +0.7598\\ +0.1480\end{bmatrix} \tag{45}$$

In accordance with Eq. (28), the lateral mode controllability matrix is

$$\mathbf{P}_2' = [\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9]' \mathbf{B}_{22} = \begin{bmatrix} -0.1422 - 1.2764i, & +0.0182 - 0.0063i \\ -0.1422 + 1.2764i, & +0.0182 + 0.0063i \\ -5.8781, & -0.1292 \\ -2.1074, & +1.2815 \\ +8.5180, & -3.6158 \end{bmatrix} \quad (46)$$

4.2 Design of Controllers for Single–Input Systems

4.2a Longitudinal modes

The set of eigenvalues (41) indicate that the uncontrolled helicopter has an unstable oscillatory mode (λ_1, λ_2) and two asymptotically stable nonoscillatory modes (λ_3, λ_4). It is evident from the mode-controllability matrix Eq. (43) that the longitudinal modes are all controllable and that, in the light of Sec. 3.1, the first three modes $(\lambda_1, \lambda_2, \lambda_3)$ would normally be controlled by U_p and the fourth mode (λ_4) by U_c.

If it is now desired to design a modal controller such that $\lambda_1 \to \rho_1$ $(= -0.8 + 1.2i)$, $\lambda_2 \to \rho_2$ $(= -0.8 - 1.2i)$, and $\lambda_3 \to \rho_3$ $(= -1.0)$ using the control input U_p, then the formula (27) implies that the required control law is

$$U_p = -6.441(u/V_R) - 1.434(w/V_R) + 0.280q + 0.586\theta \quad (47)$$

4.2b Lateral modes

The set of eigenvalues (44) indicate that the uncontrolled helicopter has an unstable oscillatory mode (λ_5, λ_6), two asymptotically stable nonoscillatory modes (λ_7, λ_8) and a neutrally stable nonoscillatory mode (λ_9). It is evident from the mode-controllability matrix (46) that the lateral modes are all controllable and that, in the light of Sec. 3.1, all the first three modes $(\lambda_5, \lambda_6, \lambda_7)$ would normally be controlled by U_R while the last two modes (λ_8, λ_9) could be controlled by either U_R or U_T.

If it is now desired to design a modal controller such that $\lambda_5 \to \rho_5$ $(= -0.5 + 1.0i)$, $\lambda_6 \to \rho_6$ $(= -0.5 - 1.0i)$, $\lambda_8 \to \rho_8$ $(= -1.0)$ and $\lambda_9 \to \rho_9$ $(= -0.6)$ using the control input U_R, then the formula (27) implies that the required control law is

$$U_R = -6.223(v/V_R) - 0.0889p - 0.628r - 0.311\phi - 0.250\psi \quad (48)$$

4.3 Sequential Design of Controllers For Multi-Input Systems

4.3a Longitudinal modes

If it is now desired to design a multi-input modal controller such that $\lambda_1 \to \rho_1$ $(= -0.8 + 1.2i)$, $\lambda_2 \to \rho_2$ $(= -0.8 - 1.2i)$ and $\lambda_3 \to \rho_3$ $(= -1.0)$ using the control U_p, and $\lambda_4 \to \rho_4$ $(= -0.6)$ using the control U_c, then the formula (27) and the procedure described in Sec. 3.3 imply that the required control laws using the sequence $\{U_p, U_c\}$ are

$$U_p = -6.441(u/V_R) - 1.434(w/V_R) + 0.280q + 0.586\theta \quad (49a)$$

and

$$U_c = -0.0014(u/V_R) + 0.651(w/V_R) - 0.000094\theta \quad (49b)$$

Alternatively, the required control laws using the sequence $\{U_c, U_p\}$ are

$$U_c = -0.00061(u/V_R) + 0.651(w/V_R) + 0.000099q - 0.000094\theta \quad (50a)$$

and

$$U_p = -6.441(u/V_R) - 1.592(w/V_R) + 0.279q + 0.586\theta \quad (50b)$$

4.3b Lateral modes

If it is now desired to design a multi-input modal controller such that $\lambda_5 \to \rho_5$ $(= -0.5 + 1.0i)$ and $\lambda_6 \to \rho_6$ $(= -0.5 - 1.0i)$, using the control U_R, and $\lambda_8 \to \rho_8$ $(= -1.0)$ and $\lambda_9 \to \rho_9$ $(= -0.6)$ using the control U_T, then the formula (27) and the procedure described in Sec. 3.3 imply that the required control laws using the sequence (U_R, U_T) are

$$U_R = -1.987(v/V_R) - 0.0441p - 0.0022r - 0.145\phi \quad (51a)$$

and

$$U_T = 1.184(v/V_R) + 0.0111p + 0.177r + 0.0439\phi + 0.0819\psi \quad (51b)$$

Alternatively, the required control laws using the sequence (U_T, U_R) are

$$U_T = 0.518(v/V_R) + 0.160p + 0.182r + 0.331\phi + 0.0812\psi \quad (52a)$$

and

$$U_R = -1.999(v/V_R) - 0.0441p - 0.0079r - 0.145\phi - 0.0022\psi \quad (52b)$$

4.4 Design of Controllers with Gain Constraints for Multi-Input Systems

The design procedure described in Sec. 3.4 can be conveniently illustrated by designing a lateral modal controller such that $\lambda_8 \to \rho_8$ $(= -0.6)$, $\lambda_9 \to \rho_9$ $(= -0.1)$ using the controls U_R and U_T. If the state variable $x_6(=p)$ is not available for feedback, then Eqs. (22) and (36) imply that the set $\{K_{ji}\}$ is unique and that the required control laws are

$$U_R = -0.507(v/V_R) + 0.244r - 0.0166\phi - 0.0194\psi \quad (53a)$$

and

$$U_T = -0.984(v/V_R) + 0.0474r - 0.0323\phi - 0.0380\psi \quad (53b)$$

Similarly, if the state variable $x_7(=r)$ is not available for feedback, the required control laws are

$$U_R = -0.427(v/V_R) - 0.0159p - 0.0452\phi - 0.0226\psi \quad (54a)$$

and

$$U_T = -0.853(v/V_R) - 0.0317p - 0.0903\phi - 0.0451\psi \quad (54b)$$

Finally, if the state variable $x_8(=\phi)$ is not available for feedback, the required control laws are

$$U_R = -0.566(v/V_R) + 0.0094p + 0.0395r - 0.0182\psi \quad (55a)$$

and

$$U_T = -1.081(v/V_T) + 0.0180p + 0.0755r - 0.0348\psi \quad (55b)$$

It is not possible to use this procedure if the state variables $x_5(=v/V_R)$ and $x_9(=\psi)$ are not available for feedback since the Eqs. (36) are linearly dependent in these cases due to the fact that the last element of $\mathbf{v}_8$ and the first element of $\mathbf{v}_9$ are both zero [(see Eqs. 45)].

5. Conclusions

The design examples presented in Sec. 4 of this paper clearly demonstrate the facility with which a system governed by a state equation of the class (1) can be designed using the procedures described in Sec. 3. This design facility results principally from the physical insight into control requirements which is obtained by the introduction of mode-controllability indices into modal control theory in the manner of Sec. 3.1. Although the design examples presented in this paper relate to autostabilizers for the control of the uncoupled longitudinal and lateral modes of an aircraft, the procedures presented in Sec. 3 may of course be used in the design of controllers in the case of aircraft with lateral-to-longitudinal and longitudinal-to-lateral inertial and aerodynamic coupling.

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